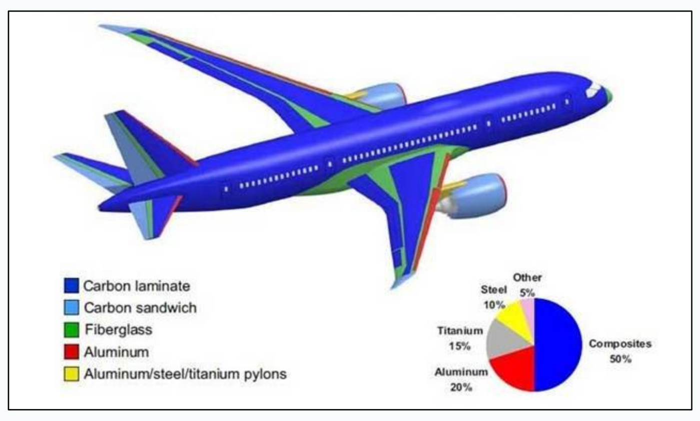
Using uncoupled and quasi-homogeneous laminates for better design of structures: on Quasi-Trivial (QT) stacking sequences and how to effectively obtain them

Candidate: *Torquato Garulli*　　**Ph. D. course in Aerospace Engineering.**　　**Supervisors:** *Anita Catapano, Daniele Fanteria, Julien Jumel*



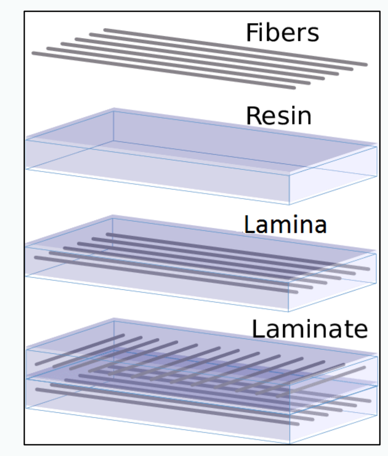
Boeing 787 materials composition. The Boeing Company ©.

Composite materials: advantages and complexity

High performance composite materials are more and more used for structural applications. While these materials could lead to **significant improvements in terms of weight reduction**, their design is not as straightforward as it is for metallic materials.
This is due, in particular, to the **anisotropic behaviour** of such materials: elastic couplings causes undesired and complex stress distributions, even in the case of extremely simple applied deformations, and vice versa.
In the aerospace industry, composite laminates are employed even for primary structures.

Composite laminates

Laminates are composed of superposed layers, in which straight and parallel fibres are dipped in a polymer matrix. Each layer has, therefore, a plane orthotropic behaviour with very good approximation. When layers are stacked, they can be oriented at any desired angle offset with respect to a fixed frame. Thus, different elastic properties can be obtained for the final laminate. Once the material (basic layer) is fixed, layers orientations are design parameters to be exploited to reach prescribed objectives.
To describe the mechanical behaviour of a laminate the easiest and most used model is Classic Laminate Plate Theory (CLPT).



Idealized representation of a composite laminate

Laminates mechanical behaviour (CLPT)

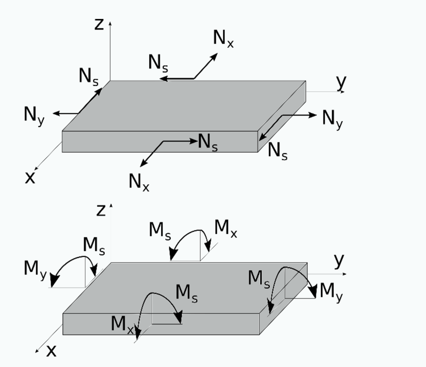
According to CLPT, laminates behaviour may be described by means of three stiffness matrices:

- Matrix **A**: describes the in-plane (membrane) behaviour;
- Matrix **B**: describes coupling between in-plane and out-of-plane behaviours;
- Matrix **D**: describes out-of-plane behaviour.

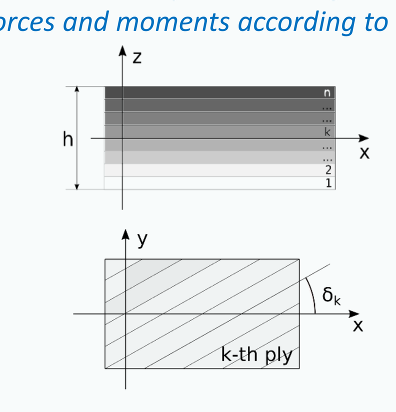
These matrices are functions of the basic properties of the material and of the stacking sequence adopted (number of plies, orientations and positions in the stack).

Obtaining uncoupling whilst also meeting other structural requirements demands a careful design of the stacking sequence.

Designers, on the other hand, use some widely known design strategies (e.g.: symmetric laminates for membrane/bending uncoupling) that extremely simplify the problem by restricting the design space to very specific classes of solutions, possibly excluding optimal solutions.



Representation of a laminate generalised forces and moments according to CLPT



Plies indexing and layers orientations definitions in a laminate

$\{N\}$= *forces per unit length*, $\{M\}$= *moments per unit length*

$$\begin{Bmatrix} N_x \\ N_y \\ N_s \\ M_x \\ M_y \\ M_s \end{Bmatrix} = \begin{bmatrix} A_{11} & A_{12} & A_{16} & B_{11} & B_{12} & B_{16} \\ A_{12} & A_{22} & A_{26} & B_{12} & B_{22} & B_{26} \\ A_{16} & A_{26} & A_{66} & B_{16} & B_{26} & B_{66} \\ B_{11} & B_{12} & B_{16} & D_{11} & D_{12} & D_{16} \\ B_{12} & B_{22} & B_{26} & D_{12} & D_{22} & D_{26} \\ B_{16} & B_{26} & B_{66} & D_{16} & D_{26} & D_{66} \end{bmatrix} \begin{Bmatrix} \varepsilon_x^0 \\ \varepsilon_y^0 \\ \varepsilon_s^0 \\ \chi_x \\ \chi_y \\ \chi_s \end{Bmatrix}$$

$\{\varepsilon^0\}$= *mid-plane strains*, $\{\chi\}$= *curvatures*

- Tension-shearing coupling (blue)
- Membrane-bending (torsion) coupling (green)
- Bending-torsion coupling (red)
- Different normalised membrane and bending behaviour (black)

STIFFNESS MATRICES

$$[A] = \frac{h}{n}\sum_{k=1}^{n}[Q(\delta_k)]$$

$$[B] = \frac{1}{2}\frac{h^2}{n^2}\sum_{k=1}^{n} b_k[Q(\delta_k)]$$

$$[D] = \frac{1}{12}\frac{h^3}{n^3}\sum_{k=1}^{n} d_k[Q(\delta_k)]$$

δ_k = kth layer orientation

$[Q(\delta_k)]$ = k^th layer reduced stiffness matrix

$$b_k = 2k - n - 1$$

$$d_k = 12k(k-n-1) + 4 + 3n(n+2)$$

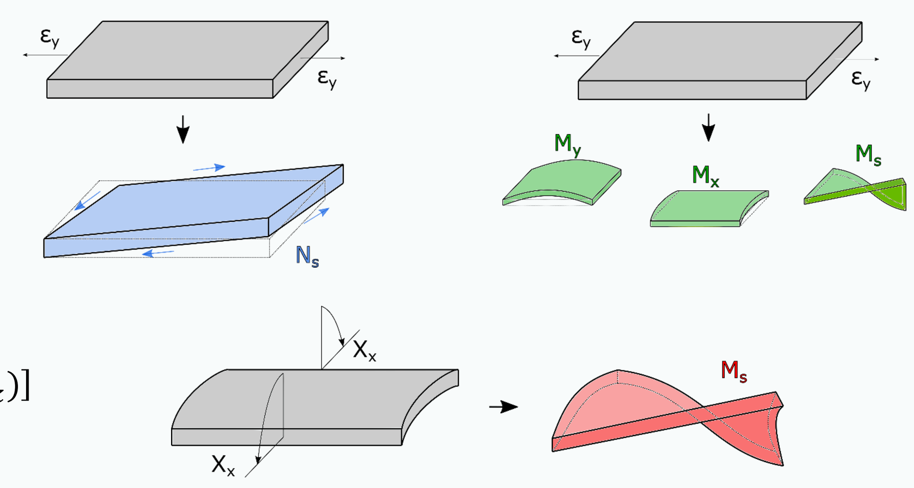
NORMALISED MATRICES

$$[A^*] = \frac{[A]}{h}, \quad [B^*] = 2\frac{[B]}{h^2}, \quad [D^*] = 12\frac{[D]}{h^3}$$

HOMOGENEITY MATRIX:

$$[C] = [A^*] - [D^*] = \frac{1}{n^3}\sum_{k=1}^{n} c_k[Q(\delta_k)]$$

$$c_k = n^2 - d_k$$



Undesired effects caused by elastic coupling terms

The concept of Quasi-Trivial solutions

Vannucci and Verchery [1] used the polar representation of CLPT stiffness matrices to reformulate conditions for uncoupling and homogeneity. They thus introduced the concept of **Quasi-Trivial solutions, which are a particular class of stacking sequences achieving uncoupling and/or homogeneity in closed-form solution**, in the framework of CLPT.

It's important to remark that **the properties of uncoupling and/or homogeneity of a QT sequence are obtained regardless of layers orientations values: only the distribution of orientations within the stack matters**. Hence, from a single QT solution infinite stacking sequences may be obtained (as orientations can be chosen freely). In the sequence proposed, for example, orientations **α**, **β**, **γ** and **δ** may assume any desired value without affecting its quasi-triviality.

Moreover, QT solutions are a relatively wide class of sequences (as an example, symmetric stacks are a very narrow subclass of uncoupled QT solutions) and hence constitute, in most cases, a sufficiently wide design space.
Thus, while uncoupling and/or homogeneity are easily achieved, layers orientations can be tailored to have the laminate meet other structural requirements (static strength, buckling load, stiffness along a given direction etc.).
Therefore, QT sequences are extremely attractive in the field of laminates design and optimization.
Why then are they not commonly adopted in composites laminates design?

1. In the literature, **only few QT stacks are available**;
2. To find QT solutions **a complex combinatorial algorithm is needed**;
3. The **length** of sequences obtainable **is limited by computational costs**.

[1] P. Vannucci, G. Verchery, *A special class of uncoupled and quasi-homogeneous laminates*, Composites Science and Technology, Volume 61, Issue 10, 2001, pp. 1465-1473.

Uncoupling $[B]=[0]$ ⟹ $\sum_{k=1}^{n} b_k e^{4i\delta_k} = 0$, $\sum_{k=1}^{n} b_k e^{2i\delta_k} = 0$

Homogeneity $[C]=[0]$ ⟹ $\sum_{k=1}^{n} c_k e^{4i\delta_k} = 0$, $\sum_{k=1}^{n} c_k e^{2i\delta_k} = 0$

Quasi-homogeneity: $[B]=[0]$, $[C]=[0]$

Theoretical proof

Consider a stacking sequence:

n = total plies number
m = number of orientations θ_j (*j=1,…,m*)

$$G_j = \{k: \delta_k = \theta_j\}$$

$$\sum_{k=1}^{n} b_k e^{4i\delta_k} =$$

$$= \sum_{k=1}^{n} b_k e^{4i\delta_k} =$$

$$= \sum_{j=1}^{m}\sum_{k\epsilon G_j} b_k e^{4i\delta_k} = \sum_{j=1}^{m} e^{4i\vartheta_j}\sum_{k\epsilon G_j} b_k$$

$$\sum_{k\epsilon G_j} \mathbf{b_k} = \mathbf{0}\ \ \forall \mathbf{j} \Rightarrow \mathbf{B} = \mathbf{0}$$

$$\sum_{k\epsilon G_j} \mathbf{c_k} = \mathbf{0}\ \ \forall \mathbf{j} \Rightarrow \mathbf{C} = \mathbf{0}$$

Example given

[α / β / γ / β / δ / α / γ / α / β]

n=9, *m*=4

G_α = {1, 6, 8}　G_β = {2, 4, 9}
G_γ = {3, 7}　G_δ = {5}

k	1	2	3	4	5	6	7	8	9
δ_k	α	β	γ	β	δ	α	γ	α	β
b_k	-8	-6	-4	-2	0	2	4	6	8

$$\sum_{k=1}^{n} b_k e^{4i\delta_k} =$$

$$= -8e^{4i\alpha} - 6e^{4i\beta} - 4e^{4i\gamma} - 2e^{4i\beta} + 0e^{4i\delta} + 2e^{4i\alpha} + 4e^{4i\gamma} + 6e^{4i\alpha} + 8e^{4i\beta} =$$

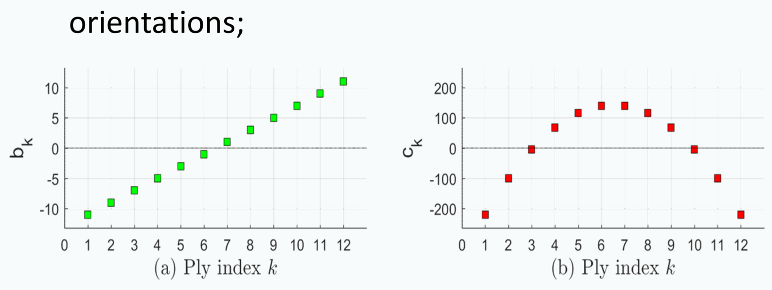
$$= e^{4i\alpha}(-8+2+6) + e^{4i\beta}(-6-2+8) + e^{4i\gamma}(-4+4) + e^{4i\delta}(0) =$$

$$= e^{4i\alpha}(0) + e^{4i\beta}(0) + e^{4i\gamma}(0) + e^{4i\delta}(0) = 0$$

Implementation of an efficient algorithm to find QT solutions

An efficient algorithm has been devised and implemented [2]. Conceptually simple algorithms may be able to find QT solutions, but they soon incur in problems related to computational costs. This limits the length of sequences that it is possible to find. The efficiency of the proposed algorithm is obtained exploiting properties of QT solutions, which are implemented in order to reduce the number of calculations to be performed. The most relevant properties exploited are:

1. **Mechanical dependence**. Consider sequences [α β γ γ β α] and [γ α β β α γ]. As orientations may assume any value, they are indeed the same sequence. Clearly it is of interest to find and store only one of the two;
2. **Mathematical dependence**. Consider sequences [α β γ γ β α] and [α β β β β α]. Again, orientations values may be chosen freely. If orientation γ is assumed equal to orientation β, the sequences become identical. As it is more general, the first sequence is the one to be stored. It is worth noting that, thanks to mathematical dependence, independent sequences with *m* orientations can be found from mathematically dependent sequences with *m-1* orientations;



Coefficients sign distribution, example of a 12 plies sequence

3. **Coefficients b_k and c_k sign distribution**. Each of these distributions allows to divide any sequence into two subsequences. As quasi-triviality requires null sums of coefficients for any given orientation, it follows that each orientation existing in the sequence must exist in each subsequence, too.

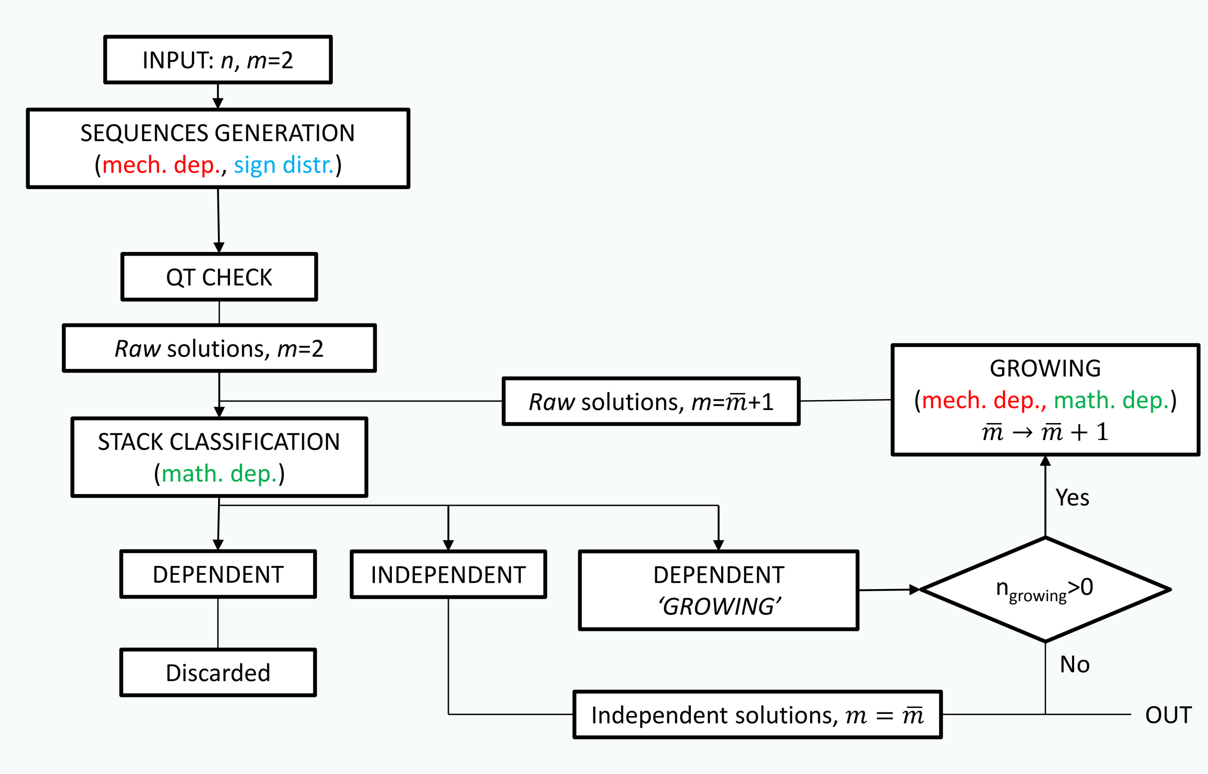
The only input to the algorithm is the desired total number of plies *n* of the sequences to be found. The number of orientations *m* is fixed to 2. Independent QT solutions with more of orientations are found later exploiting mathematical dependence.

Initially, sequences of the chosen length are generated. The algorithm exploits mechanical dependence and sign distributions of coefficients b_k and c_k to avoid generating sequences that are known, a priori, not to be QT. This increase the efficience.
Then, sequences generated are checked for quasi-triviality and a set of raw (including dependent) solutions with two orientations is obtained.
This set of solutions is the input to a loop working as follows: sequences are separated into: dependent ones, which are discarded; independent ones, which are stored; dependent growing ones. If the number of growing sequences is not zero, a new raw set of QT solution with high number of orientations is obtained from them and used as new input for the loop. Thus the loop goes on until the maximum number of orientations allowing the existence of QT solutions is reached.

Thanks to this algorithm two very important results have been reached:

1. **Longer sequences have been found, with respect to previous studies;**
2. **For fixed total plies number and number of orientations, more solutions have been found with respect to previous studies.**

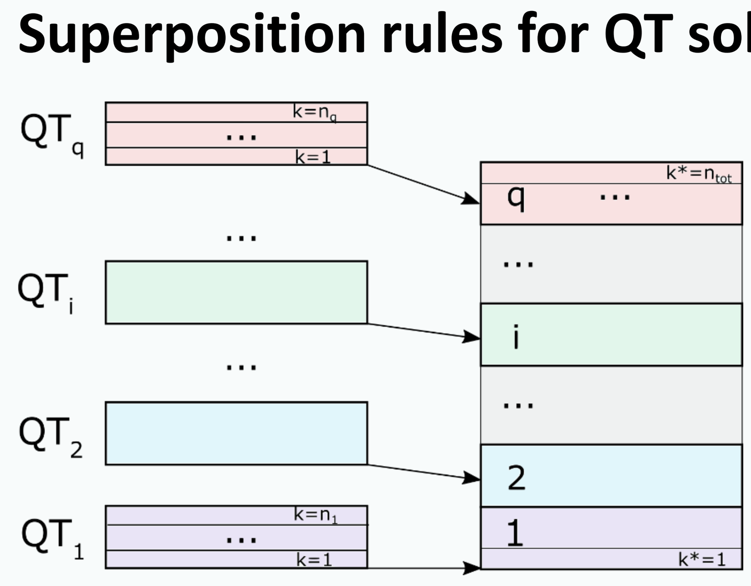
Thus, an extended database of QT solutions has been created.



Workflow of the QT solutions finder algorithm

[2] T. Garulli, A. Catapano, M. Montemurro, J. Jumel, D. Fanteria, *Quasi-trivial solutions for uncoupled, homogeneous and quasi-homogeneous laminates with high number of plies*, ECCM 6, 11-15 June 2018, Glasgow.

Superposition rules for QT solutions



Despite the results presented above, the maximum length of QT solutions obtainable is still limited by computational costs. This clearly represents a limit to the adoption of QT sequences. In order to overcome this limit, a series of **closed-form analytical rules to obtain new QT solutions by superposition of known ones have been derived** [3]. These conditions apply to **any number of superposed sequences**, with any number of orientations and express conditions that each and every orientation group of the superposed sequences should respect for the macro-sequence to be a QT one.

The procedure to derive these rules consists in the following steps:

1. A number *q* of known QT solutions is superposed;
2. The equations of uncoupling and/or homogeneity are written for the macro-sequence resulting from the superposition;
3. Transformation equations are used to express these conditions in terms of parameters of the initial sequences;
4. Quasi-triviality of the initial sequences is exploited to simplify the equations;
5. Algebraic development.

Thanks to such rules, QT solution of any desired total plies number may be obtained.

[3] T. Garulli, A. Catapano, M. Montemurro, J. Jumel, D. Fanteria, *Quasi-trivial stacking sequences for the design of thick laminates*, Composites Structures, Vol. 200, pages 614-623, September 2018.

Conclusion

QT solutions could facilitate and improve laminates design, by eliminating coupling problems.
Their adoption, however, is still very limited, due to little knowledge and lack of tools to obtain them.
Here, an efficient algorithm to find such solutions has been presented.
Additionally, superposition rules have been derived.

These tools should prove extremely useful to adopt QT solutions in laminate design. In particular they allow, for the first time, to find QT sequences of any desired length.
Hence, laminates with a high number of plies (e.g.: laminates using *thin plies*) may be conceived using such sequence.